Some Inclusion Relationships of Certain Subclasses of Analytic Functions Defined by Komatu Integral Operator.

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Abstract- **The authors introduce several new subclasses of analytic functions. In this paper, we recall some notations and standard definitions of analytic functions in the open unit disc. We state the definition of the Hadamard product (or convolution).We introduce and study some new subclasses of analytic functions, starlike function, convex function, close-to-convex function and quasi-convex function. We introduce and investigate some new subclasses of uniformly starlike. uniformly convex univalent functions. Also, in this paper we state the definition of Komatu integral operator and discuss the use of integral operator. After that, we explain what a new subclass of uniformly convex functions and new subclass of uniformly starlike functions are. Finally, we state the definitions and lemmas concerning the subordinations we need in our discussion. In this paper, we establish Inclusion Relationships for uniformly certain classes of analytic functions, associated with the aforementioned integral operator.**

Keywords- Analytic functions; Univalent functions; Starlike functions; Convex functions; Subordination; Uniformly functions; Integral operator.

I. INTRODUCTION AND DEFINITIONS

Let A denote the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

which are analytic in the open unit disc $U = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$,
Let $f \in A$ given by (1) and $g \in A$ given by

$$\mathrm{g(z) = z + \sum_{k=2}^{\infty} b_k z^k} \qquad (2)$$

the Hadamard product (*or convolution*) of f and g is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k\, z^k = (g * f)(z) \qquad (3)$$

Given two functions f and g, which are analytic in U with $f(0) = g(0)$

The function f is said to ***subordinate*** to g in U if there exists a function w, analytic in U, such that

$w(0) = 0,\ \ |w(z) < 1\ \ (z \in U),$ and
$f(z) = g\big(w(z)\big)\ \ (z \in U).$

We denote this subordination by

$$f(z) \prec g(z)\ \ in\ U.$$

If g is univalent, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)\ \ and\ f(U) \subset g(z).$$

Further, Let S denote the subclass of all functions in A consisting of ***univalent functions*** in U.

Definition 1:[1],[2]. A function $f(z) \in S$ is said to be ***starlike function*** of order α $(0 \le \alpha < 1)$, if and only if

$$Re\left\{\frac{zf`(z)}{f(z)}\right\} > \alpha \quad (z \in U). \qquad (4)$$

We denote by $S^*(\alpha)$ the class of all starlike function of order α in U.

Also, a function $f(z)$ belonging to S is said to be ***convex function*** of order α $(0 \le \alpha < 1)$, if and only if

$$Re\left\{\frac{zf`(z)}{f(z)}\right\} > \alpha \quad (z \in U). \qquad (5)$$

We denote by $K(\alpha)$ the class of all convex functions of order α .

The classes $S^*(\alpha)$ and $K(\alpha)$ are related to each other as follows:

$$f(z) \in K(\alpha) \quad \Leftrightarrow \quad zf'(z) \in S^*(\alpha) \qquad (6)$$

We note that $S^*(0) = S^*$ and $K(0) = K$

Also, the classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by [1], [2] and studied by [3], [4].

A function $f(z) \in S$ is said to be ***close-to-convex function*** of order α, denote by $C(\alpha)$ if and only if

$$Re\left\{\frac{zf`(z)}{g(z)}\right\} > \alpha \quad (0 \le \alpha 1; g \in S^*; z \in U). \qquad (7)$$

We denote by $C(\alpha)$ the class of all close-to-convex functions of order α (see [5]).

We note that $C(0) = C$ is the class of all close-to-convex functions (see [6]) and a function $f \in S$ is said to be ***quasi-convex function*** of order α $(0 \le \alpha < 1)$, if it satisfies

$$Re\left\{\frac{(zf`(z))``}{g`(z)}\right\} > \alpha \ \ (z \in U) \qquad (8)$$

For some $g \in K$. We denode by $C^*(\alpha)$ the class of all

quasi-convex functions of order α (see [7]).

Received-August 2020, Accepted-October 2020, and Published-December 2020

Definition 2:[1], [8], [9], [10]. A function $f(z)$ of the form (1) is said to be in the class of ***uniformly starlike functions***, denoted by $\boldsymbol{UST}$, if it satisfies the following condition:

$$Re\left\{\frac{zf`(z)}{f(z)}\right\} > \left|\frac{zf`(z)}{f(z)} - 1\right| \quad (z \in U). \tag{9}$$

Definition 3:[1], [8], [11], [12]. A function $f(z)$ of the form (1) is said to be in the class of ***uniformly convex functions***, denoted by $\boldsymbol{UCV}$, if it satisfies the following condition:

$$Re\left\{1 + \frac{zf``(z)}{f`(z)}\right\} > \left|\frac{zf``(z)}{f`(z)}\right| \quad (z \in U). \tag{10}$$

From (9) and (10), it follows that:

$$f(z) \in UCV \Leftrightarrow zf`(z) \in UST. \tag{11}$$

Recently, Komatu ([13], [14]) introduced a certain integral operator $L_a^\lambda(a > 0;\ \lambda \geq 0)$ defined by

$$L_a^\lambda f(z) = \frac{a^\lambda}{\Gamma(\lambda)}\int_0^1 t^{a-2}\left(log\frac{1}{t}\right)^{\lambda-1} f(zt)dt \tag{12}$$

$$(z \in U; a > 0;\ \lambda \geq 0; f(z) \in A)$$

Thus, if $f(z) \in A$ is of the form (1), it is easily seen from (12) that(see[15])

$$L_a^\lambda f(z) = z + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^\lambda a_n z^n \tag{13}$$

$$(a > 0; \lambda \geq 0)$$

Using the above relation, it is to simple verify that

$$z\left(L_a^{\lambda+1}f(z)\right)` = aL_a^\lambda f(z) - (a-1)L_a^{\lambda+1}f(z) \tag{14}$$

$$(a > 0; \lambda \geq 0)$$

We note that:

(i) For $a = 1$ and $\lambda = k$ $(k\ is\ any\ integer)$, the multiplier transformation $L_1^\lambda f(z) = I^k f(z)$ was studied by([15], [16])

(ii) For $a = 1$ and $\lambda = -k$ $(k \in N_0 = \{0,1,2,\dots\}$, the differential operator $L_1^{-k} f(z) = D^k f(z)$ was studied by [16];

(iii) For $a = 2$ and $\lambda = k$ $(k\ is\ any\ integer)$, the operator $L_2^\lambda f(z) = L^k f(z)$ was studied by [17]

(iv) For $a = 2$, the multiplier transformation $L_2^\lambda f(z) = I^\lambda f(z)$ was studied by [18].

Now, Let f be given by (1) and the integral operator $\boldsymbol{L_a^\lambda}: \boldsymbol{A} \to \boldsymbol{A}$ be given by (13) , (see [1]).
we define new classes by using the operator $\boldsymbol{L_a^\lambda}$ as follows:

Definition 4. Let $f(z) \in \boldsymbol{A}$. Then $f(z) \in SP_a^\lambda(\beta, \gamma)$ if and only if

$$L_a^\lambda f(z) \in SP(\beta, \gamma). \tag{15}$$

Definition 5. Let $f(z) \in \boldsymbol{A}$. Then $f(z) \in UCV_a^\lambda(\beta, \gamma)$ if and only if

$$L_a^\lambda f(z) \in UCV(\beta, \gamma). \tag{16}$$

Definition 6. Let $f(z) \in \boldsymbol{A}$. Then $f(z) \in UKC_a^\lambda(\beta, \gamma)$ if and only if

$$L_a^\lambda f(z) \in UKC(\beta, \gamma). \tag{17}$$

Definition 7. Let $f(z) \in \boldsymbol{A}$. Then $f(z) \in UQC_a^\lambda(\beta, \gamma)$ if and only if

$$L_a^\lambda f(z) \in UQC(\beta, \gamma). \tag{18}$$

We not that

$$f(z) \in UCV_a^\lambda(\beta, \gamma) \Leftrightarrow zf`(z) \in SP_a^\lambda(\beta, \gamma) \tag{19}$$

$$f(z) \in UQC_a^\lambda(\beta, \gamma) \Leftrightarrow zf`(z) \in UKC_a^\lambda(\beta, \gamma) \tag{20}$$

II- THE MAIN INCLUSION RELATIONSHIPS

In order to prove our main results, we shall require the following lemmas.

Lemma 1: [1], [19],[20]. Let h be univalent convex function in U with $h(0) = c$ and $Re(ah(z) + b) > 0,\ (a, b \in \mathbb{C})$.

Let $p(z) = c + \sum_{n=1}^{\infty} b_n z^n$ be analytic in U. Then $P(z) + \frac{zp`(z)}{ap(z)+b} \prec h(z)\ in\ U.$

Implies that $P(z) \prec h(z)\ \ in\ U.$

Lemma 2: [1], [19]. Let h be convex function in U and let $D > 0$. Suppose also that $Q(z)$ is analytic in U with $Re\{Q(z)\} > D,\ (z \in U)$. If p(z) be analytic in U and $p(0) = h(0)$. Then

$$P(z) + Q(z)zp`(z) \prec h(z)\ \ in\ U.$$

Implies that $P(z) \prec h(z)\ in\ U.$

Theorem 1:- $SP_a^\lambda(\beta, \gamma) \subset SP_a^{\lambda+1}(\beta, \gamma)$,
$(f(z) \in A; a > 0, \beta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0)$

Proof:- Let $f(z) \in SP_a^\lambda(\beta, \gamma)$ and suppose that

$$\frac{z(L_a^{\lambda+1}f(z))`}{L_a^{\lambda+1}f(z)} = P(z) \tag{21}$$

Where $P(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in U and $P(z) \neq 0$ for all $z \in U$.
Using the identity (14), We have

$$\frac{aL_a^\lambda f(z)}{L_a^{\lambda+1}f(z)} = P(z) + (a-1) \tag{22}$$

By using the logarithmic differentiation on both sides of (22) with respect to z, we obtain

$$\frac{z(L_a^\lambda f(z))`}{L_a^\lambda f(z)} = \frac{z(L_a^{\lambda+1}f(z))`}{L_a^{\lambda+1}f(z)} + \frac{zp`(z)}{p(z)+(a-1)}$$

Which, in view of (21), yields

$$\frac{z(L_a^\lambda f(z))`}{L_a^\lambda f(z)} = P(z) + \frac{zP`(z)}{P(z)+(a-1)} \tag{23}$$

From (23) we see that

$Re\{h(z) + (a-1)\} > 0, \quad (z \in U),$ and
$P(z) + \frac{zP`(z)}{P(z)+(a-1)} \prec h(z)$,

Thus, by using lemma 1 and (21), we observe that

Received-August 2020, Accepted-October 2020, and Published-December 2020

$$P(z) \prec h(z),$$

So that

$$f(z) \in SP_a^{\lambda+1}(\beta,\gamma),$$

This implies that

$$SP_a^{\lambda}(\beta,\gamma) \subset SP_a^{\lambda+1}(\beta,\gamma).$$

This completes the proof of Theorem 1.

Theorem 2:- $UCV_a^{\lambda}(\beta,\gamma) \subset UCV_a^{\lambda+1}(\beta,\gamma)$.
$(f(z) \in A; a > 0, \beta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0)$.
Proof:

$$\begin{aligned} f(z) \in UCV_a^{\lambda}(\beta,\gamma) &\Leftrightarrow L_a^{\lambda} f(z) \in UCV(\beta,\gamma) \\ &\Leftrightarrow z\left(L_a^{\lambda} f(z)\right)' \in SP(\beta,\gamma) \\ &\Leftrightarrow L_a^{\lambda}(zf'(z)) \in SP(\beta,\gamma) \\ &\Leftrightarrow zf'(z) \in SP_a^{\lambda}(\beta,\gamma) \\ &\Rightarrow zf'(z) \in SP_a^{\lambda+1}(\beta,\gamma) \\ &\Leftrightarrow zf'(z) \in SP_a^{\lambda}(\beta,\gamma) \\ &\Rightarrow zf'(z) \in SP_a^{\lambda+1}(\beta,\gamma) \\ &\Leftrightarrow L_a^{\lambda+1}\left(zf'(z)\right) \in SP(\beta,\gamma) \\ &\Leftrightarrow z(L_a^{\lambda+1} f(z))' \in SP(\beta,\gamma) \\ &\Leftrightarrow L_a^{\lambda+1} f(z) \in UCV(\beta,\gamma) \\ &\Leftrightarrow f(z) \in UCV_a^{\lambda+1}(\beta,\gamma) \end{aligned}$$

Which evidently proves Theorem 2.

Theorem 3:- $UKC_a^{\lambda}(\beta,\gamma) \subset UKC_a^{\lambda+1}(\beta,\gamma)$,
$(f(z) \in A; a > 0, \beta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0)$
Proof:- Let $f(z) \in UKC_a^{\lambda}(\beta,\gamma)$.
Then, by (17), there exists a function
$q(z) \in SP(\beta,\gamma)$ $(0 \leq \gamma < 1)$ such that

$$\frac{z(L_a^{\lambda} f(z))`}{q(z)} \prec \psi(z) \quad in\ U \tag{24}$$

Taking

$$q(z) = L_a^{\lambda} g(z) \in \mathrm{SP}(\beta,\gamma) \quad (0 \leq \gamma < 1) \tag{25}$$

We find from (15) and (25) that
$g(z) \in SP_a^{\lambda}(\beta,\gamma)$ and

$$\frac{z(L_a^{\lambda} f(z))`}{L_a^{\lambda} g(z)} \prec \psi(z) \quad in\ U$$

Respectively. We now Let

$$\frac{z(L_a^{\lambda+1} f(z))`}{L_a^{\lambda+1} g(z)} = P(z) \tag{26}$$

Where $P(z) = 1 + p_1 z + p_2 z^2 + \cdots$
Making use of the operator identity (14), we also have

$$\frac{z(L_a^{\lambda} f(z))`}{L_a^{\lambda} g(z)} = \frac{L_a^{\lambda}(zf`(z))}{L_a^{\lambda} g(z)}$$
$$= \frac{z\left(L_a^{\lambda+1}\left(zf'(z)\right)\right)` + (a-1)L_a^{\lambda+1}(zf`(z))}{z\left(L_a^{\lambda+1} g(z)\right)` + (a-1)L_a^{\lambda+1} g(z)}$$

By theorem 1, we know that

$$g(z) \in SP_a^{\lambda}(\beta,\gamma) \text{ and } SP_a^{\lambda}(\beta,\gamma) \subset SP_a^{\lambda+1}(\beta,\gamma) \tag{27}$$

So that we can put

$$\frac{z(L_a^{\lambda+1} g(z))`}{L_a^{\lambda+1} g(z)} = G(z) \tag{28}$$

Where $Re\{G(z)\} > 0$, $(z \in U)$

Upon substituting from (26) and (28) into (27), we have

$$\frac{z(L_a^{\lambda} f(z))'}{L_a^{\lambda} g(z)} = \frac{\dfrac{z\left(L_a^{\lambda+1}\left(zf`(z)\right)\right)`}{L_a^{\lambda+1} g(z)} + (a-1)\dfrac{L_a^{\lambda+1}(zf`(z))}{L_a^{\lambda+1} g(z)}}{\dfrac{z\left(L_a^{\lambda+1} g(z)\right)`}{L_a^{\lambda+1} g(z)} + (a-1)}$$
$$= \frac{\dfrac{z\left(L_a^{\lambda+1}\left(zf`(z)\right)\right)`}{L_a^{\lambda+1} g(z)} + (a-1)P(z)}{G(z) + (a-1)} \tag{29}$$

Now from (26) , we have

$$z\left(L_a^{\lambda+1} f(z)\right)` = P(z) L_a^{\lambda+1} g(z). \tag{30}$$

Differentiating both sides of (30) with respect to z, we have

$$\begin{aligned} \left(z\left(L_a^{\lambda+1} f(z)\right)`\right)` = L_a^{\lambda+1} g(z)\left(zP`(z)\right) \\ + P(z) z(L_a^{\lambda+1} g(z))` \end{aligned}$$
$$\frac{z\left(L_a^{\lambda+1}\left(zf`(z)\right)\right)`}{L_a^{\lambda+1} g(z)} = G(z)P(z) + zP`(z) \tag{31}$$

Making use of (24),(29) and (31), we get

$$\frac{z(L_a^{\lambda} f(z))`}{L_a^{\lambda} g(z)} = P(z) + \frac{zP`(z)}{G(z) + (a-1)} \prec \psi(z) \tag{32}$$

Since $Re\{G(z) + a - 1\} > 0$, $(z \in U)$
Hence, be taking

$$Q(z) = \frac{1}{G(z) + (a-1)}$$

In (32), and applying Lemma 2, we can show that

$$P(z) \prec \psi(z) \quad in\ U$$

So that

$$f(z) \in UKC_a^{\lambda+1}(\beta,\gamma)$$

This implies that

$$UKC_a^{\lambda}(\beta,\gamma) \subset UKC_a^{\lambda+1}(\beta,\gamma)$$

This completes the proof of theorem 3.

Theorem 4:- $UQC_a^{\lambda}(\beta,\gamma) \subset UQC_a^{\lambda+1}(\beta,\gamma)$.
$(f(z) \in A; a > 0, \beta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0)$.

Proof:-

$$\begin{aligned} f(z) \in UQC_a^{\lambda}(\beta,\gamma) &\Leftrightarrow L_a^{\lambda} f(z) \in UQC(\beta,\gamma) \\ &\Leftrightarrow z\left(L_a^{\lambda} f(z)\right)` \in UKC(\beta,\gamma) \\ &\Leftrightarrow L_a^{\lambda}(zf`(z)) \in UKC(\beta,\gamma) \\ &\Leftrightarrow zf`(z) \in UKC(\beta,\gamma) \\ &\Rightarrow zf`(z) \in UKC_a^{\lambda+1}(\beta,\gamma) \\ &\Leftrightarrow L_a^{\lambda+1}\left(zf`(z)\right) \in UKC(\beta,\gamma) \\ &\Leftrightarrow z(L_a^{\lambda+1} f(z))` \in UKC(\beta,\gamma) \\ &\Leftrightarrow L_a^{\lambda+1} f(z) \in UQC(\beta,\gamma) \\ &\Leftrightarrow f(z) \in UQC_a^{\lambda+1}(\beta,\gamma) \end{aligned}$$

Which evidently proves Theorem 4.

Received-August 2020, Accepted-October 2020, and Published-December 2020

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Received-August 2020, Accepted-October 2020, and Published-December 2020